PERIOD-V

M10CH8

TOPIC

8 Simultaneous Linear Equations

8.1 SIMULTANEOUS LINEAR EQUATIONS

Simultaneous linear equations are a pair of equations which are true (solved) at the same time i.e. simultaneously. If we have two linear equations and we wish to make both equations true at the same time, we require values for the variables which satisfy both the equations. These values are the *simaltaneous solutions* to the pair of equations.

Consider a pair of *simultaneous linear equations* containing two unknowns, usually x and y. There are infinitely many values of x and y which satisfy the *first equation*. Likewise, there are infinitely many values of x and y which satisfy the *second equation*. In general, however, only one combination of values of x and y satisfies *both* the equations at the same time.

For example: Consider the pair of simultaneous linear equations

$$\begin{cases} x + y = 9 \\ 2x + 3y = 21 \end{cases}$$

If $x = 6$ and $y = 3$ then

- $x + y = 6 + 3 = 9$. The first equation is satisfied.
- $2x + 3y = 2(6) + 3(3) = 12 + 9 = 21$. The second equation is also satisfied.

So, $x = 6$ and $y = 3$ is the *solution* to the simultaneous equations

$$x + y = 9 \quad \text{and} \quad 2x + 3y = 21.$$

Example 1. *Find the simultaneous solution to the equations:*

$$y = 2x - 1 \quad \text{and} \quad y = x + 3.$$

Solution. If $y = 2x - 1$ and $y = x + 3$, then

$$2x - 1 = x + 3 \qquad \text{(Equating } y\text{'s)}$$

$$\therefore \quad 2x - 1 - x = x + 3 - x \qquad \text{(Subtracting } x \text{ from both sides)}$$

$\therefore \quad x - 1 = 3$

$\therefore \quad x = 4$ (Adding 1 to both sides)

and so $y = 4 + 3$ (Using $y = x + 3$)

$\therefore \quad y = 7$

So, the simultaneous solution is $x = 4$ and $y = 7$.

Check: In $y = 2x - 1$, $y = 2 \times 4 - 1 = 8 - 1 = 7$

In $y = x + 3$, $y = 4 + 3 = 7$

8.2 TRUTH SET FOR SIMULTANEOUS LINEAR RELATIONS

Solving an equation is the process of finding the truth values of the equation. The set of all truth values of an equation is the truth set (solution set) of the equation.

Example 2. *Consider the following simultaneous linear equations and find their common set of truth values: $2y - x = 8$ and $y - 2x = 1$.*

Solution. We have $2y - x = 8$

By subject changing formula, we get

Arbitrary values $\quad x = 2y - 8$ ...(1)

When $y = 2$, $x = 2(2) - 8 = 4 - 8 = -4$

When $y = 3$, $x = 2(3) - 8 = 6 - 8 = -2$

When $y = 5$, $x = 2(5) - 8 = 10 - 8 = 2$

} Fixed values

and so on.

$\therefore$ Truth set of equation (1) = $\{(-4, 2), (-2, 3), (2, 5)\} = S_1$ (say)

Similarly, for $y - 2x = 1$ ...(2)

Arbitrary values $\quad y = 1 + 2x$

When $x = -1$, $y = 1 + 2(-1) = 1 - 2 = -1$

When $x = 0$, $y = 1 + 2(0) = 1$

When $x = 2$, $y = 1 + 2(2) = 1 + 4 = 5$

} Fixed values

and so on.

$\therefore$ Truth set of equation (2) = $\{(-1, -1), (0, 1), (2, 5)\} = S_2$ (say)

Now, $S_1 \cap S_2 = \{(2, 5)\}$ (Singleton set)

Which is the common set of truth values of the given simultaneous equations.

Example 3. *Find the truth set of the following equations: $x + y = 3$ and $3x - 2y = 4$. Also find their common set of truth values.*

Solution. For $x + y = 3$

By subject changing formula, we get

Arbitrary values		
	$y = 3 - x$	...(1)
When $x = 1$,	$y = 3 - 1 = 2$	Fixed values
When $x = 2$,	$y = 3 - 2 = 1$	
When $x = 3$,	$y = 3 - 3 = 0$	

and so on

∴ Truth set of equation (1),

$$S_1 = \{(1, 2), (2, 1), (3, 0)\}$$

Similarly, for $3x - 2y = 4$, we have

$$2y = 3x - 4 \Rightarrow y = \frac{3x-4}{2} \qquad ...(2)$$

Arbitrary values		
When $x = 0$,	$y = \frac{3(0)-4}{2} = \frac{-4}{2} = -2$	Fixed values
When $x = 4$,	$y = \frac{3(4)-4}{2} = \frac{8}{2} = 4$	
When $x = 2$,	$y = \frac{3(2)-4}{2} = \frac{6-4}{2} = \frac{2}{2} = 1$	

and so on.

Note: You may find more truth set but you can stop when you find out the common set.

∴ Truth set of equation (2), $S_2 = \{(0, -2), (4, 4), (2, 1)\}$

Now, $S_1 \cap S_2 = \{(2, 1)\}$, which is the required truth set that is singleton.

EXERCISE 8.1

1. Find the simultaneous solution to the following pairs of equations:

(*a*) $y = x - 2$, $y = 3x + 6$ (*b*) $y = x + 2$, $y = 2x - 3$ (*c*) $y = 6x - 6$, $y = x + 4$ (*d*) $y = 2x + 1$, $y = x - 3$

2. Consider the following simultaneous linear equations and find their common set of truth values.

(*a*) $x + 3y = 6$; $2x - 3y = 12$ (*b*) $2x - y - 4 = 0$; $x + y + 1 = 0$

(*c*) $2x - y = 2$; $4x + 3y = 24$

8.3 FINDING SOLUTION(S) USING GRAPHS

Two linear equations in two variables form *a system of linear equations.* A *solution* to a system of linear equations is an ordered pair which satisfies both equations in the system.

For example: The ordered pair (3, 4) satisfies the system of equations

	$2x - 3y + 6 = 0$	and	$2x - y - 2 = 0$
i.e.,	$2 \times 3 - 3 \times 4 + 6 = 0$	*i.e.,*	$2 \times 3 - 4 - 2 = 0$
or	$6 - 12 + 6 = 0$	or	$6 - 6 = 0$

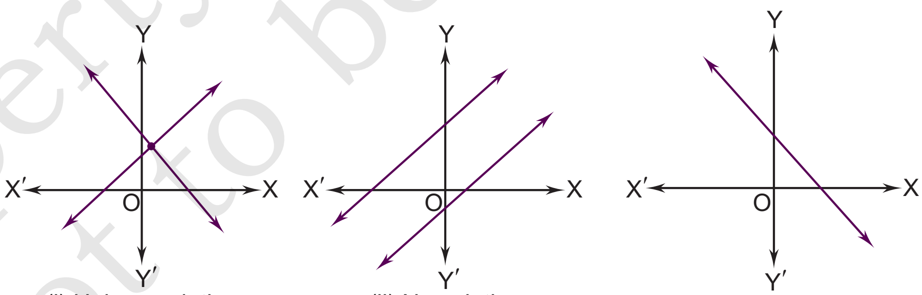
both are true statements.

Thus, $x = 3$ and $y = 4$ or the ordered pair (3, 4) is a solution. The solution set to the system is {(3, 4)}.

The solution can be obtained by graphing both equations. The coordinates of the point of intersection give the solution set of the system.

If the lines intersect, they will intersect in only one point giving a unique solution to the system as shown in the figure (*i*) If the lines are parallel, there is no point of intersection and, hence, no solution (see figure (*ii*)).

If the lines are coincident, *i.e.,* the same line for both equations, then every point on the line is a common point and, hence, there are infinitely many solutions (see figure (*iii*)).



(i) Unique solution (intersecting lines) (ii) No solution (parallel lines) (iii) Infinitely many solutions (coincident lines)

We shall now concentrate on the system having a unique solution.

Example 4. *Find the solution set of the following system of equations graphically:* $2x + 5y = 10$ *and* $x = -5$.

Solution. Given equations are:

$$2x + 5y = 10 \quad \Rightarrow \quad y = \frac{10 - 2x}{5} \quad \ldots(1) \qquad x = -5 \quad \ldots(2)$$

Table of values for (1)

x	0	5	→ Arbitrary
y	2	0	→ Fixed

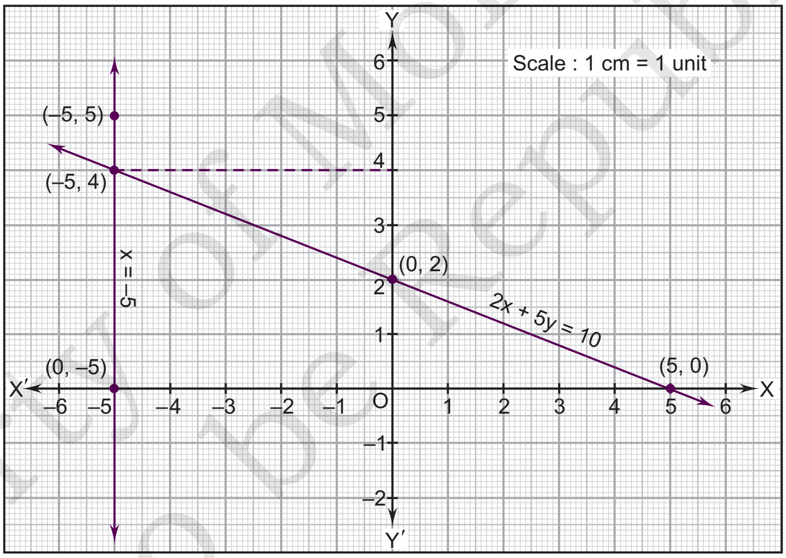
Table of values for (2)

x	–5	–5	→ Arbitrary
y	0	5	→ Fixed

Plot the ordered pairs (0, 2) and (5, 0). Join and produce both ways. This is the graph of equation (1).

Plot the ordered pairs (–5, 0) and (–5, 5). Join and produce both ways. This vertical line is the graph of equation (2).

The two lines intersect at a unique point (–5, 4) (as shown in the figure). Therefore, the ordered pair (–5, 4), *i.e.*, $x = -5$, $y = 4$ is the solution of the system. The solution set $S = \{(-5, 4)\}$ is a singleton set.



Example 5. *Find the solution set of the following system of equations graphically:* $2x - y - 1 = 0$ *and* $x - 2y + 1 = 0$.

Solution. The given equations can be written as

$$y = 2x - 1 \quad \ldots(1) \qquad y = \frac{x + 1}{2} \quad \ldots(2)$$

Table of values for (1)

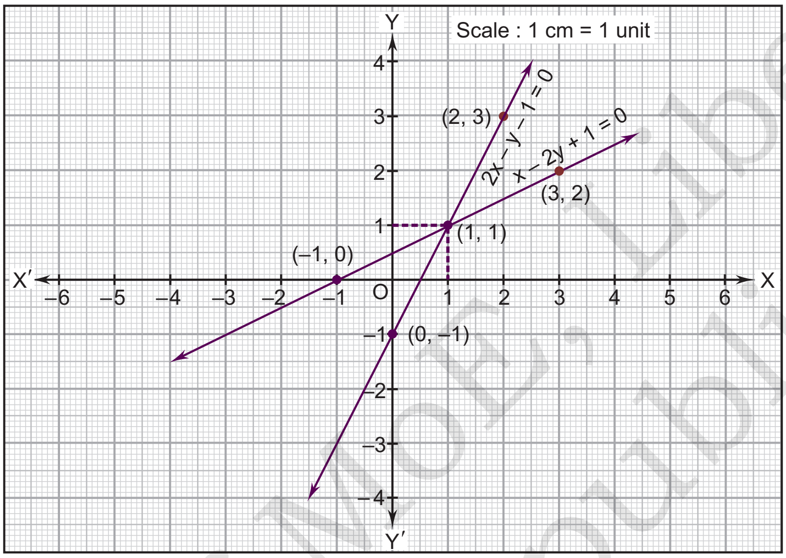
x	0	2
y	–1	3

Table of values for (2)

x	–1	3
y	0	2

Plot the ordered pairs (0, –1) and (2, 3). Join and produce both ways. This is the graph of equation (1).

Plot the ordered pairs (–1, 0) and (3, 2). Join and produce both ways. This is the graph of equation (2) (see the figure given below).



The two lines intersect at a unique point (1, 1). Therefore, the solution set is S = {(1, 1)}.

EXERCISE 8.2

1. Find the solution set of the following system of equations graphically: $2x - 3y = 1$, $3x - 4y = 1$.
2. Solve the following pair of linear equations graphically: $2x + y - 6 = 0$, $4x - 2y - 4 = 0$.
3. Draw the graph of the equations $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Using this graph, find the values of x and y which satisfy both the equations.

8.4 ELIMINATION METHOD

Under this method, we seek to make one of the two variables becomes *zero* and then solve for the *other variable* as in linear equation in one variable. Note the following steps.

1. Make the coefficient of the variable you wish to eliminate the same (*i.e.* to have the same absolute value).

2. If the values of both coefficients are positive or negative, then *subtract*, but if one is positive and the other is negative, then *add* the equations.
3. Solve the resulting equation and find the value of its variable.
4. Substitute (or replace) the value of the variable obtained in *step 3* above into one of the original equations and solve for the remaining variable.

Note: If the equations are mixed up, re-arrange them before solving.

Example 6. *Solve the following simultaneous equations:*

(*i*) $x + y = 7,\ 2x - y = 5$ (*ii*) $\frac{1}{2}a + \frac{1}{3}b = 1,\ \frac{1}{4}a - \frac{1}{2}b = -2$

Solution.

(*i*)
$$x + y = 7 \quad \ldots(1)$$
$$+\ 2x - y = 5 \quad \ldots(2)$$
$$3x + 0y = 12$$

[Adding (1) and (2) in order to eliminate y]

$$\Rightarrow \quad 3x = 12 \quad \Rightarrow \quad x = \frac{12}{3} = 4$$

Replacing $x = 4$ in (1), we have

$$4 + y = 7 \quad \Rightarrow \quad y = 7 - 4 = 3 \quad \Rightarrow \quad y = 3$$

$\therefore$ The solution is $x = 4,\ y = 3$.

(*ii*)
$$\frac{1}{2}a + \frac{1}{3}b = 1 \quad \ldots(1)$$
$$\frac{1}{4}a - \frac{1}{2}b = -2 \quad \ldots(2)$$

Multiplying (1) by the LCM(2, 3) = 6 and (2) by the LCM (2, 4) = 4,

(1) becomes $3a + 2b = 6$ …(3)
(2) becomes $a - 2b = -8$ …(4)
} Adding in order to eliminate y

$$4a = -2 \quad \Rightarrow \quad a = \frac{-2}{4} = -\frac{1}{2}$$

Replacing $a = -\frac{1}{2}$ in (1), $\frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{3}b = 1$

$$\Rightarrow \quad -\frac{1}{4} + \frac{1}{3}b = 1 \quad \Rightarrow \quad \frac{1}{3}b = 1 + \frac{1}{4}$$

$$\Rightarrow \quad \frac{1}{3}b = \frac{5}{4} \quad \Rightarrow \quad b = \frac{3 \times 5}{4} = \frac{15}{4} = 3\frac{3}{4}$$

$\therefore$ The solution is $a = -\frac{1}{2}$ and $b = 3\frac{3}{4}$.

8.5 SUBSTITUTION METHOD

To solve a pair of simultaneous linear equations by the substitution method, the following steps should be taken.

1. Make one of the variables the subject in one of the equations and substitute it into the *other* equation.
2. Solve for the variable in the equation obtained in step one.
3. Substitute the value of the variable you have solved for in step two above into one of the original equations and solve for the other variables.

Example 7. *Solve the following system of equations:*

(*i*) $x + 3y = 8,\ x + y = 2$ (*ii*) $5x - 3y = 30,\ x = -3y - 12$

Solution. (*i*) $x + 3y = 8$ ...(1)

$x + y = 2$ ...(2)

Making x the subject in (2) and substituting it in (1),

(2) becomes $x = 2 - y$

Substituting $x = 2 - y$ in (1),

$2 - y + 3y = 8 \Rightarrow 2y = 8 - 2$

$\Rightarrow 2y = 6 \Rightarrow y = 3$

Substituting $y = 3$ in (2),

$\Rightarrow x + 3 = 2 \Rightarrow x = 2 - 3$

$\therefore x = -1$

Therefore the solution is: $x = -1,\ y = 3$.

(*ii*) $5x - 3y = 30$ ...(1)

$x = -3y - 12$ ...(2)

Substituting equation (2) in equation (1),

$5(-3y - 12) - 3y = 30$

$\Rightarrow -15y - 60 - 3y = 30 \Rightarrow -18y = 90$

$$\Rightarrow \frac{-18y}{-18} = \frac{90}{-18} \Rightarrow y = -5$$

Substituting $y = -5$ in equation (2),

$x = -3(-5) - 12 \Rightarrow x = 15 - 12$

$\therefore x = 3$

Therefore the solution is: $x = 3,\ y = -5$.

EXERCISE 8.3

1. Solve each of the following pairs of linear equations by the *Elimination Method:*

(*a*) $2x - y = 6;\ x - y = 2$ (*b*) $x + y = 6;\ x - y = 2$

(*c*) $7x - 8y - 11 = 0;\ 8x - 7y - 7 = 0$ (*d*) $2x + 7y - 11 = 0;\ 3x - y - 5 = 0$

2. Solve each of the following pairs of linear equations by the *Elimination Method*:

(*a*) $4x + \frac{y}{3} = \frac{8}{3};\ \frac{x}{2} + \frac{3y}{4} = -\frac{5}{2}$ (*b*) $\frac{3}{x} - 5y + 1 = 0;\ \frac{2}{x} - y + 3 = 0$

(*c*) $\frac{4}{x} + 3y = 14;\ \frac{3}{x} - 4y = 23.$

3. Solve each of the following pairs of linear equations by the *Elimination Method:*

(*a*) $47x + 31y = 63;\ 31x + 47y = 15$

(*b*) $37x + 41y = 70;\ 41x + 37y = 86.$

4. Solve the following pair of linear equations for x and y, by the *Substitution Method*:

(*a*) $7x - 15y = 2;\ x + 2y = 3$ (*b*) $2x + 3y = 9;\ 3x + 4y = 5$

(*c*) $x - y = 0;\ 2x - y = 2$ (*d*) $2x - y = 2;\ 3x - 4y = -2$

5. Solve for x and y by the *Substitution Method*:

(*a*) $2x = 5y + 4;\ 3x - 2y + 16 = 0$ (*b*) $2x - y + 3 = 0;\ 3x - 5y + 1 = 0$

8.6 WORD PROBLEMS ON SIMULTANEOUS LINEAR EQUATIONS

To solve a word problem involving two unknown quantities:

(*i*) Assume the two unknown quantities as x and y.

(*ii*) Using given conditions construct two linear equations in two variables x and y.

(*iii*) Solve the pair of linear equations simultaneously to get the values of x and y.

Example 8. *A family of three adults and two children paid L\$ 72 as the fare for a journey. Another family of four adults and three children paid L\$ 99 as the fare for the same journey. Calculate the fare for*

(*i*) *an adult* (*ii*) *a child*

Solution. Let the fare for an adult be L\$ x and for a child be L\$ y.

Given:

(*i*) Fare for three adults (= L$ $3x$) and two children (= L$ $2y$) is L$ 72

$$\Rightarrow \quad 3x + 2y = 72 \quad \ldots(1)$$

(*ii*) Fare for four adults (= L$ $4x$) and three children (= L$ $3y$) is L$ 99

$$\Rightarrow \quad 4x + 3y = 99 \quad \ldots(2)$$

To make the coefficient of y equal in the two equations, we multiply (1) by 3 and (2) by 2, getting

$$9x + 6y = 216 \quad \ldots(3)$$

and

$$8x + 6y = 198 \quad \ldots(4)$$

Subtracting (4) from (3), we get $x = 18$

Replacing x by 18 in equation (1), we get

$$3 \times 18 + 2y = 72 \quad \Rightarrow \quad 2y = 72 - 54$$

$$\Rightarrow \quad 2y = 18 \quad \Rightarrow \quad y = 9$$

Therefore,

(*i*) the fare for an adult (x) = L$ 18

(*ii*) the fare for a child (y) = L$ 9

Example 9. *A number consists of two digits whose sum is 8. When 18 is added to the number, the digits are reversed. Find the number.*

Solution. Let x be the digit in ten's place and y be the digit in unit's place. Then the number is $10x + y$. When the digits are reversed, we have y in ten's place and x in unit's place so that the new number is $10y + x$.

Given: (*i*) Sum of digits is 8

$$\Rightarrow \quad x + y = 8 \quad \ldots(1)$$

(*ii*) Original number + 18 = New number

$$\Rightarrow \quad (10x + y) + 18 = 10y + x$$

$$\Rightarrow \quad 10x - x + y - 10y = -18$$

$$\Rightarrow \quad 9x - 9y = -18$$

Dividing throughout by 9, we have

$$x - y = -2 \quad \ldots(2)$$

Adding (1) and (2), we get

$$2x = 6 \quad \Rightarrow \quad x = \frac{6}{2} = 3$$

Replacing x by 3 in (1), we have

$$3 + y = 8 \quad \Rightarrow \quad y = 8 - 3 \quad \Rightarrow \quad y = 5$$

Therefore, the required number is 35.

Example 10. *Five years ago, father's age was six times his son's age. Five years hence, their ages will be in the ratio 8 : 3. Find their present ages.*

Solution. Let father's present age be x years and son's present age be y years.

Five years ago, their ages were $(x - 5)$ years and $(y - 5)$ years.

Given: $x - 5 = 6(y - 5) \quad \Rightarrow \quad x - 5 = 6y - 30$

$\Rightarrow \quad x - 6y - 5 + 30 = 0 \quad \Rightarrow x - 6y + 25 = 0$ ...(1)

Five years hence, their ages will be $(x + 5)$ years and $(y + 5)$ years.

Given: $\dfrac{x+5}{y+5} = \dfrac{8}{3} \quad \Rightarrow \quad 3(x + 5) = 8(y + 5)$

$\Rightarrow \quad 3x + 15 = 8y + 40 \Rightarrow 3x - 8y + 15 - 40 = 0$

$\Rightarrow \quad 3x - 8y - 25 = 0$ ...(2)

Multiplying (1) by 3, we have

$3x - 18y + 75 = 0$ ...(3)

Subtracting (3) from (2),

$$10y - 100 = 0 \quad \text{or} \quad 10y = 100$$

$$\Rightarrow \quad y = \frac{100}{10} = 10$$

Putting $y = 10$ in (2), we have

$3x - 8 \times 10 - 25 = 0$

$\Rightarrow \quad 3x - 80 - 25 = 0 \quad \Rightarrow \quad 3x - 105 = 0$

$\Rightarrow \quad 3x = 105 \quad \Rightarrow \quad x = \dfrac{105}{3} = 35$

Hence, father's present age = 35 years and son's present age = 10 years.

EXERCISE 8.4

1. The sum of two numbers is 29. Their difference is 17. Find the two numbers.
2. One pencil and 2 erasers cost L$ 8. Two pencils and 3 erasers cost L$ 14. How much does each pencil and each eraser cost?

3. The sum of the ages of Emine and Samuel is 29 years. Emine is 3 years older than Samuel. How old are they?
4. Tickets for a film were sold L$ 450 to the general public and L$ 375 to pupils. 400 people attended the show and L$ 168000 was collected in ticket sales.
   (*a*) How many tickets were sold to pupils?
   (*b*) Mr. Samuel was issued with 25 tickets to be sold to the general public and 20 tickets to be sold to pupils. How much did Mr. Samuel collect after selling the tickets issued to him?
5. The sum of two numbers is 8 and their product is –33. Find the two numbers.
6. The total age of two sisters is 108 years. One is 18 years older than the other. Find the ratio of the age of the older to the younger.

REVIEW EXERCISE

1. Find the simultaneous solution of the following pairs of equations:
   (*a*) $y = x + 4$, $y = 5 - x$ (*b*) $y = x + 1$, $y = 7 - x$ (*c*) $y = 2x - 5$, $y = 3 - x$ (*d*) $y = x - 4$, $y = -2x - 4$
2. Consider the following simultaneous linear equations and find their common set of truth values.
   (*a*) $2x + y + 6 = 0$; $2x - y + 2 = 0$ (*b*) $x - y = 1$; $2x + y = 8$
3. Find the solution set of the following system of equations graphically:
   $3x - 4y + 6 = 0$, $3x + y - 9 = 0$.
4. Solve the following pair of linear equations graphically:
   $x + 3y = 6$; $2x - 3y = 12$.
5. Solve each of the following pairs of linear equations by the *Elimination Method*:
   (*a*) $\frac{4}{x} + 5y = 7$; $\frac{3}{x} + 4y = 5$ (*b*) $x + \frac{6}{y} = 6$; $3x - \frac{8}{y} = 5$
6. Solve each of the following pairs of linear equations by the *Elimination Method:*
   (*a*) $31x + 13y = 57$; $13x + 31y = 75$ (*b*) $37x + 43y = 123$; $43x + 37y = 117$
7. Solve the following pair of linear equations for x and y, by the *Substitution Method*:
   (*a*) $3x - 5y = -1$; $x - y = -1$ (*b*) $x + 2y = -1$; $2x - 3y = 12$
8. Solve for x and y by the *Substitution Method*:

$$3x + 4y = 13;\ 2x - 3y = 3$$

9. If the numerator is increased by 2 and the denominator is decreased by 3, a fraction equals one. If the numerator is decreased by 3 and the denominator increased by 4, the fraction becomes $\frac{1}{4}$. Find the fraction.

10. Mr. Daniel and Mr. Felix run a small business assembling two types of product. The cost of components and the labour needed for each product is shown in the table below.

	Cost of component	Labour man-hours
Type A	36	16
Type B	24	24

The business has L$ 156 available to buy components each week. The total labour available each week is 96 man-hours. How many products of each type can they assemble each week to maintain maximum production?

MULTIPLE CHOICE QUESTIONS (MCQs)

1. The system of a pair of simultaneous linear equations $x = 0$, $y = 3$ has

(a) no solution (b) a unique solution

(c) two solutions (d) infinitely many solutions

2. The pair of simultaneous equations $x = 4$ and $y = -3$ graphically represent lines which are

(a) coincident (b) parallel

(c) intersecting at (4, –3) (d) intersecting at (–3, 4)

3. If $2x + 3y = 13$ and $5x - 4y = -2$, then $x + y$ equals

(a) 6 (b) –6 (c) 5 (d) –5

4. The solution of the pair of simultaneous linear equations $29x + 37y - 103 = 0$, $37x + 29y - 95 = 0$, is

(a) $x = 1, y = 2$ (b) $x = 2, y = 1$ (c) $x = 2, y = 3$ (d) $x = 3, y = 2$

5. If the pair of simultaneous linear equations $2x - y - 3 = 0$, $2kx + 7y - 5 = 0$ has a unique solution $x = 1$, $y = -1$ then the value of k is

(a) 3 (b) 4 (c) 6 (d) –6

6. The pair of simultaneous linear equations $x + 2y - 5 = 0$, $7x + 3y - 13 = 0$ has a unique solution. Then, the values of x and y are

(a) $x = 1, y = 2$ (b) $x = 2, y = 1$ (c) $x = 3, y = 1$ (d) $x = 1, y = 3$

7. A number consists of two digits, whose sum is 10. If 18 is subtracted from the number, digits are reversed. Then the number is

(a) 73 (b) 64 (c) 55 (d) None of these

8. If the sum of the ages of a mother and her son in years is 65, and twice the difference their ages in years is 50, then the age of the mother is

(a) 45 years (b) 40 years (c) 50 years (d) None of these

RECAP AT A GLANCE

- Simultaneous linear equations are a pair of equations which are true (solved) at the same time i.e. simultaneously.
- Solving an equation is the process of finding the truth values of the equation.
- Two linear equations in two variables form *a system of linear equations.*
- If the equations are mixed up, re-arrange them before solving.

❑ ❑ ❑